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Quantum Field Theory in Black Hole-Type Spacetimes with Horizons

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Using quantum field theory in black hole-type spacetimes with horizons, which includes all the black hole solutions and also some other interesting solutions in general relativity, we obtain Hawking's thermal spectrum of Dirac particles near the event horizon as well as the cosmological horizon of the spacetime.

1. INTRODUCTION

Hawking's investigation of quantum effects (Hawking, 1974) interpreted as the emission of a thermal spectrum of particles by a black hole event horizon has been extended by Gibbons and Hawking (1977) to spacetimes with cosmological event horizon. The de Sitter spacetime with cosmological event horizon has attracted renewed interest as a model of the inflationary scenario of the early universe. The thermal radiation in the spacetimes with event horizon as well as cosmological horizons has been studied by various authors (Ahmed, 1991; Dianyan and Huiya, 1985; Gen, 1985). The thermal radiation in NUT-Kerr-Newman-de Sitter spacetime (Ahmed, 1991) is interesting in that the thermal radiation is possible even in the NUT spacetime, which may be thought unphysical.

In this paper we study the thermal radiation in the spacetime with magnetic monopoles which possesses an event horizon as well as a cosmological horizon. This study will be interesting in that reasons to believe magnetic monopoles exist have been given on the grounds of the symmetry that they would introduce in the field equations of electromagnetism. This monopole hypothesis was propounded by Dirac relatively long ago. The ingenious suggestion by Dirac that the magnetic monopole does exist was neglected

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due to the failure to detect such a particle. However, in recent years the development of gauge theories have shed new light on this.

2. THE BACKGROUND SPACETIME

We consider the spacetime

$$ds^2 = \frac{\Sigma}{\Delta_\theta} d\theta^2 + \frac{\Sigma}{\Delta_r} dr^2 + \frac{E^{-2}\Delta_\theta \sin^2\theta}{\Sigma}(a\, dt - \rho\, d\Phi)^2 + \frac{E^{-2}\Delta_r}{\Sigma}(dt - A\, d\Phi)^2 \qquad (1)$$

where

$$\Sigma = r^2 + (n + a\cos\theta)^2$$

$$\Delta_\theta = 1 + \frac{\Lambda}{3} a^2 \cos^2\theta$$

$$\Delta_r = (r^2 + a^2 + n^2)\left[1 - \frac{\Lambda}{3}(r^2 + 5n^2)\right] - 2(Mr + n^2) + e^2 + g^2$$

$$E = 1 + \frac{\Lambda}{3} a^2$$

$$\rho = r^2 + a^2 + n^2$$

$$A = a\sin^2\theta - 2n\cos\theta$$

Besides the cosmological constant Λ, the spacetime given by (1) possesses five real parameters: the mass parameter M, the NUT (magnetic mass) parameter n, the angular momentum per unit mass parameter a, the electric charge parameter e, and the magnetic monopole parameter g.

We call the metric given by (1) the NUT-Kerr-Newman-de Sitter metric generalized with the magnetic monopole. Further, we may call the metric the generalized hot NUT-Kerr-Newman (GHNUTKN) metric, since the cosmological (de Sitter) parameter has been interpreted as being hot (Gasperini, 1988). For $\Lambda = n = g = 0$ the GHNUTKN metric reduces to the well-known Kerr–Newman metric.

3. DECOUPLED EQUATION

Kamran and McLenaghan (1984) obtained the separation of the Dirac equation in a general background. From Kamran and McLenaghan's equation in the proper limit, we obtain the radial decoupled Dirac equation for the electron in the GHNUTKN spacetime as follows:

$$\Delta_r \frac{d^2R(r)}{dr^2} + \left[\sqrt{\Delta_r}\frac{d}{dr}(\sqrt{\Delta_r}) - \frac{im\Delta_r}{\lambda + imr}\right]\frac{dR(r)}{dr}$$

$$+ \left[E^2K^2\Delta_r^{-1} - \lambda^2 - m^2r^2 + \frac{mEK}{\lambda + imr} + iE\sqrt{\Delta_r}\frac{d}{dr}\left(\frac{K}{\sqrt{\Delta_r}}\right)\right]R(r) = 0$$

$$K = r^2\omega - \alpha - \frac{eQr}{E} \tag{2}$$

where ω is the energy of the Dirac particle, α is its projected angular momentum, λ is the separation constant, and m and Q are the mass and the electric charge of the Dirac particles, respectively.

With the coordinate transformation

$$\frac{d}{d\hat{r}} = \frac{\Delta_r}{\beta}\frac{d}{dr} \tag{3}$$

where

$$\beta = r^2 + (a + n)^2$$

equation (2) can be reduced to

$$\beta^2 \frac{d^2R}{d\hat{r}^2} + \left(2r\Delta_r - \beta\frac{\Delta_r'}{2} - \frac{im\beta\Delta_r}{\lambda + imr}\right)\frac{dR}{d\hat{r}}$$

$$+ \Delta_r\left(\frac{E^2K^2}{\Delta_r} - \lambda^2 - m^2r^2 + \frac{mEK}{\lambda + imr} - \frac{iEK\Delta_r'}{2\Delta_r}\right.$$

$$\left. + i2Er\omega - ieQ\right)R = 0 \tag{4}$$

where the prime denotes differentiation with respect to the argument.

Near the horizons $\Delta_r = 0$, equation (4) reduces to

$$\frac{d^2R}{d\hat{r}^2} + E^2\left(\frac{K^2}{\beta^2}\right)R = 0 \tag{5}$$

Near $r = r_+$, the solution of the wave equation (5) can easily be found to be

$$R \sim \exp[\pm iE(\omega - \omega_0)]\ \hat{r} \tag{6}$$

where (Cabibbo and Ferrari, 1962; Rohrlich, 1966)

$$\omega_0 = \frac{(a + n)^2\omega + \alpha}{r_+^2 + (a + n)^2} + \frac{eQr_+}{E[r_+^2 + (a + n)^2]} \tag{7}$$

and r_+, called the event horizon, is the smaller of the two positive values of r at which $\Delta_r = 0$, provided the roots are real (i.e., $1/\Lambda > M^2 > a^2 - n^2 + e^2 + g^2$). The larger positive value of $\Delta_r = 0$ denoted by r_{++} represents the cosmological horizon.

Now we can write the radial wave function as

$$\Psi_r = \exp[-i\omega(t \pm \hat{r}_1)] \tag{8}$$

where

$$\hat{r}_1 = \frac{E(\omega - \omega_0)\hat{r}}{\omega} \tag{9}$$

We resolve Ψ_r into ingoing and outgoing waves as

$$\Psi_r^{\text{in}} \sim \exp[-i\omega(t + \hat{r}_1)] \tag{10}$$

$$\Psi_r^{\text{out}} \sim \exp[-i\omega(t - \hat{r}_1)] \tag{11}$$

Introducing the Eddington coordinates

$$V = t + \hat{r}_1 \tag{12}$$

we obtain

$$\Psi_r^{\text{in}} \sim \exp(-i\omega v) \tag{13}$$

$$\Psi_r^{\text{out}} \sim \exp[-i\omega v + 2iE(\omega - \omega_0)\hat{r}] \tag{14}$$

Near $r = r_+$, equation (3) can be integrated to give

$$\hat{r} = \frac{1}{2E\kappa_+} \ln(r - r_+) \tag{15}$$

where

$$\kappa_+ = -\frac{\Lambda}{6E[r_+^2 + (a + n)^2]} (r_+ - r_{++})(r_+ - r_-)(r_+ - r_{--}) \tag{16}$$

is the surface gravity of the event horizon of the GHNUTKN spacetime and r_-, r_{--} are the other two roots of $\Delta_r = 0$. Just outside the event horizon

$$\Psi_r^{\text{out}} \sim e^{-i\omega v}(r - r_+)^{(i/\kappa_+)(\omega-\omega_0)} \tag{17}$$

We now extend the outgoing wave outside the horizon to the region inside. Since on the event horizon, the outgoing wave function is not analytic and cannot be straightforwardly extended to the region inside, it can be continued analytically to the complex plane by going around the horizon. We go along the lower semicircle of radius $|r - r_+|$, where the variable is

$$|r - r_+|e^{-i\pi} = (r_+ - r)e^{-i\pi}$$

Hence inside the event horizon

$$\Psi_r^{\text{out}} \sim e^{-i\omega v}(r_+ - r)^{(i/\kappa_+)(\omega-\omega_0)}e^{(\pi/\kappa_+)(\omega-\omega_0)} \tag{18}$$

Introducing the step function

$$y(x) = \begin{cases} 1 & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x < 0 \end{cases} \tag{19}$$

we can generally write the outgoing wave function as

$$\begin{aligned} \Phi_r^{\text{out}} = N_r\{ & y(r - r_+)\Psi_r^{\text{out}}(r - r_+) \\ & + y(r_+ - r)\Psi_r^{\text{out}}(r_+ - r)\exp\{(\pi/\kappa_+)(\omega - \omega_0)\} \end{aligned} \tag{20}$$

where Ψ_r^{out} is the normalized Dirac wave function.

Expression (20) describes the splitting of Φ_r^{out} into two components:

(a) A flow of positive-energy particles of strength N_r^2 outgoing from the event horizon.

(b) A flow of positive-energy particles propagating in the GHNUTKN background gravitational field in reverse time, since inside the event horizon, r represents the time axis due to the interchange of time and space. This can be interpreted as a flow in time of negative-energy antiparticles ingoing toward the singularity region. This shows that the wave function near the event horizon is such that there is creation of particle–antiparticle pairs (Deruelle and Ruffini 1975a,b).

Obviously, from the normalization condition, we have

$$\langle\Phi_r^{\text{out}}, \Phi_r^{\text{out}}\rangle = N_r^2\{\exp[(2\pi/\kappa_+)(\omega - \omega_0)] + 1\} = 1 \tag{21}$$

or

$$\begin{aligned} N_r^2 &= \{\exp[(2\pi/\kappa_+)(\omega - \omega_0)] + 1\}^{-1} \\ &= \{\exp[(1/\kappa_B T_+)(\omega - \omega_0)] + 1\}^{-1} \end{aligned} \tag{22}$$

where

$$T_+ = \frac{\kappa_+}{2\pi\kappa_B} \tag{23}$$

T_+ is the temperature of the region inside the event horizon; κ_B is Boltzmann's constant. Equation (22) is the main formula demonstrating the emission of a thermal spectrum of Dirac particles by the event horizon of the GHNUTKN spacetime.

Following in a similar way, we have

$$T_{++} = \frac{\kappa_{++}}{2\pi\kappa_B} \tag{24}$$

where

$$\kappa_{++} = -\frac{\Lambda}{6E[r_{++}^2 + (a + n)^2]}(r_{++} - r_+)(r_{++} - r_-)(r_{++} - r_{--}) \tag{25}$$

is the surface gravity of the cosmological horizon.

4. DISCUSSION

In this study we observe that Hawking's thermal spectrum can be found in spacetimes with magnetic monopoles provided the horizons exist. But in the case that the number of magnetic monopoles in a celestial body of the order of galactic nuclei may be very large and can be compared with the gravitational mass of the source in the gravitational effect (Deyu *et al.*, 1984; Qiuhe and Yongjiu, 1985; Yongjiu and Qiuhe, 1985) there actually exist no horizons. Thus we conclude that Hawking radiation in this spacetime is possible only if the mass of the celestial body is greater than the mass of the monopoles present.

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